TRANSFORMATION GEOMETRY

LESSON 8

In Grade 11, you studied rotations about the origin through an angle of 90° in a clockwise or anticlockwise direction. Rotations through an angle of 180° about the origin were also studied.

In Grade 12 we need to be able to find the image of a point that is rotated about the origin through any angle θ, in a clockwise or anticlockwise direction.

The formulae:

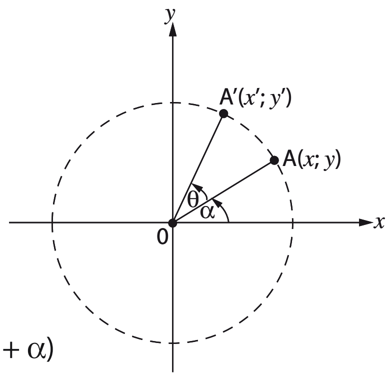
Rotation of point A(x; y) through an angle of θ degrees in an anticlockwise direction:

$$x' = x_A \cos\theta - y_A \sin\theta$$
$$y' = y_A \cos\theta + x_A \sin\theta$$

Rotation through an angle of θ degrees in a clockwise direction:

Replace θ with (–θ)

Where do the formulae come from?



Study the diagram on the right. The rotation from A to A′, about the origin, means that the distance from O to A is the same as the distance from O to A′.

Let us say that OA = OA′ = r

From the trig definitions,

$\frac{x}{r} = \cos\alpha$; $\frac{y}{r} = \sin\alpha$; $\frac{x'}{r} = \cos(\theta + \alpha)$; $\frac{y'}{r} = \sin(\theta + \alpha)$

$\frac{x'}{r} = \cos\theta.\cos\alpha - \sin\theta\sin\alpha$

$\therefore \frac{x'}{r} = \cos\theta.\frac{x}{r} - \sin\theta.\frac{y}{r}$

$\therefore x' = r\left(\cos\theta.\frac{x}{r} - \sin\theta.\frac{y}{r}\right)$

$\therefore x' = x\cos\theta - y\sin\theta$

$\frac{y'}{r} = \sin\theta.\cos\alpha + \sin\alpha.\cos\theta$

$\therefore \frac{y'}{r} = \sin\theta.\frac{x}{r} + \frac{y}{r}.\cos\theta$

$\therefore y' = r\left(\sin\theta.\frac{x}{r} + \frac{y}{r}.\cos\theta\right)$

$\therefore y' = x\sin\theta + y\cos\theta = y\cos\theta + x\sin\theta$

Applications of the formula

Example

Example 1

Determine, without using a calculator, the image A′ obtained when the point A(3; 4) is rotated about the origin through an angle of 30° in:

1. a clockwise direction
2. an anticlockwise direction

Solution

Solution

1. $x' = x_A\cos\theta - y_A\sin\theta = 3\cos 30° - 4\sin 30° = 3\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} - 2$

   $y' = y_A\cos\theta + x_A\sin\theta = 4\cos 30° + 3\sin 30° = 4\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{1}{2}\right) = 2\sqrt{3} + \frac{3}{2}$

   $\therefore A'\left(\frac{3\sqrt{3} - 4}{2}; \frac{4\sqrt{3} + 3}{2}\right)$

2. $x' = x_A \cos\theta - y_A \sin\theta$

$= 3\cos(-30°) - 4\sin(-30°)$

$= 3\cos 30° + 4\sin 30°$

$= 3\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)$

$= 3\frac{\sqrt{3}}{2} + 2$

$\therefore A'\left(\frac{3\sqrt{3}+4}{2}; \frac{4\sqrt{3}-3}{2}\right)$

$y' = y_A \cos\theta + x_A \sin\theta$

$= 4\cos(-30°) + 3\sin(-30°)$

$= 4\cos 30° - 3\sin 30°$

$= 4\left(\frac{\sqrt{3}}{2}\right) - 3\left(\frac{1}{2}\right)$

$= 2\sqrt{3} - \frac{3}{2}$

Example 2

Point T(2; 2) is rotated about the origin, O, through an acute angle of θ in an anticlockwise direction. The image point is given by T′ $(1 - \sqrt{3}; y)$; $y > 0$.

Determine: 1. the value of y

2. the angle θ.

Solution

1. OT = OT′ rotation preserved length

$\therefore (2-0)^2 + (2-0)^2 = (1 - \sqrt{3} - 0)^2 + (y - 0)^2$ (using the distance formula)

$\therefore 8 = 1 - 2\sqrt{3} + 3 + y^2$

$\therefore y^2 = 4 + 2\sqrt{3}$

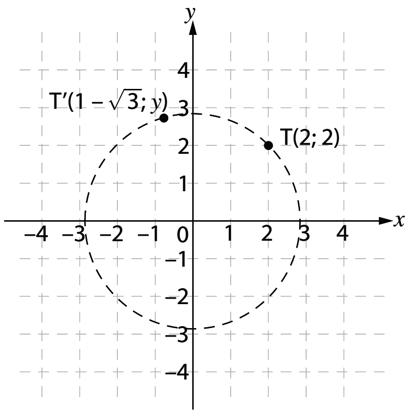
$\therefore y^2 = (1 + \sqrt{3})^2$

$\therefore y = 1 + \sqrt{3}$

Note: $4 + 2\sqrt{3}$

$= 1 + 2\sqrt{3} + (\sqrt{3})^2$

$= (1 + \sqrt{3})^2$



2. $x' = x_T \cos\theta - y_T \sin\theta$

$\therefore 2\cos\theta - 2\sin\theta = 1 - \sqrt{3} \ldots$(A)

$y' = y_T \cos\theta + x_T \sin\theta$

$\therefore 2\cos\theta + 2\sin\theta = 1 + \sqrt{3} \ldots$(B)

(A) + (B):

$4\cos\theta = 2$

$\therefore \cos\theta = \frac{1}{2}$

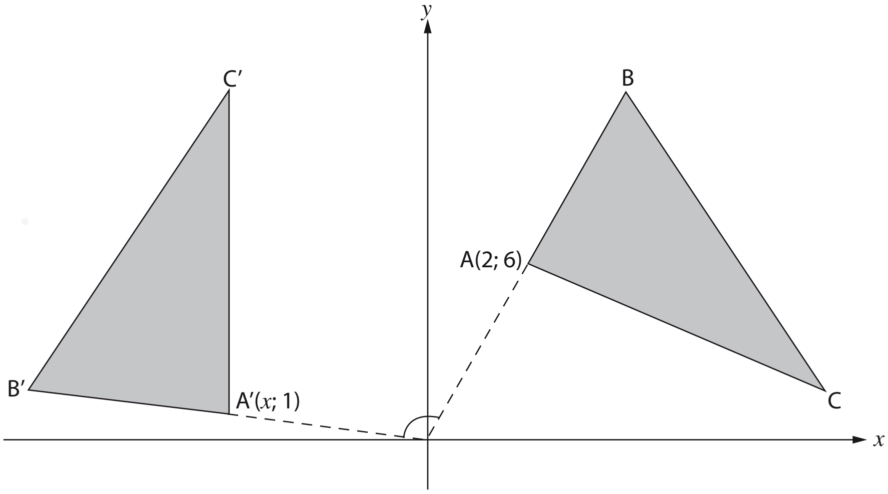
$\therefore \theta = 60°$

Example 3

In the diagram on the next page triangle A′B′C′ is the image of triangle ABC after a rotation of θ° about the origin.

If the coordinates of A and A′ are (2; 6) and (x; 1) respectively, determine

1. the value of x
2. the value of θ



Solution

Solutions

1. $OA = OA'$: rotation about the origin

$\therefore (2-0)^2 + (6-0)^2 = (x-0)^2 + (1-0)^2$

$\therefore 4 + 36 = x^2 + 1$

$\therefore x^2 = 39$

$\therefore x = -\sqrt{39}$

2. $x' = x\cos\theta - y\sin\theta$ $\qquad$ $y' = y\cos\theta + x\sin\theta$

$\therefore -\sqrt{39} = 2\cos\theta - 6\sin\theta \ldots A$ $\qquad$ $\therefore 1 = 6\cos\theta + 2\sin\theta \ldots B$

Multiplying B by 3, gives $18\cos\theta + 6\sin\theta = 3 \ldots C$

C + A yields: $20\cos\theta = -\sqrt{39} + 3$

$\therefore \cos\theta = -0{,}162249\ldots$

$\therefore \theta = 180° - 80{,}7° = 99{,}3°$

Example

Example 4

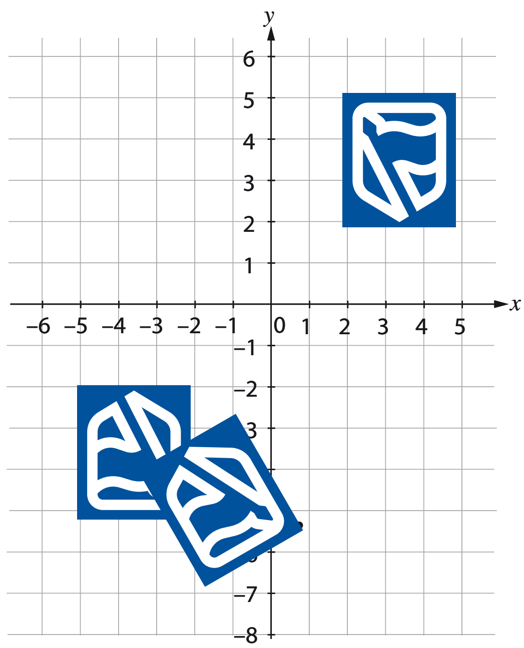
The logo of a well known and popular bank in South Africa, namely Standard Bank, is placed in the Cartesian Plane.

The logo in the first quadrant is rotated through 180° about the origin and then it is further rotated through 30°in an anticlockwise direction about the origin.

Determine the image point of A(4,5 ; 2,5) after

1. the first rotation, and
2. the second rotation.

Give answers correct to two decimal places.



Solutions

Solution

1. Rule for rotation through 180°: $(x; y) \rightarrow (-x; -y)$

   Therefore, A′(−4,5; −2,5)

2. Formula for rotation in anticlockwise direction:

   $x' = x \cos\theta - y \sin\theta$

   $y' = y \cos\theta + x \sin\theta$

   $x' = -4{,}5.\cos 30° - (-2{,}5).\sin 30° = -2{,}6$

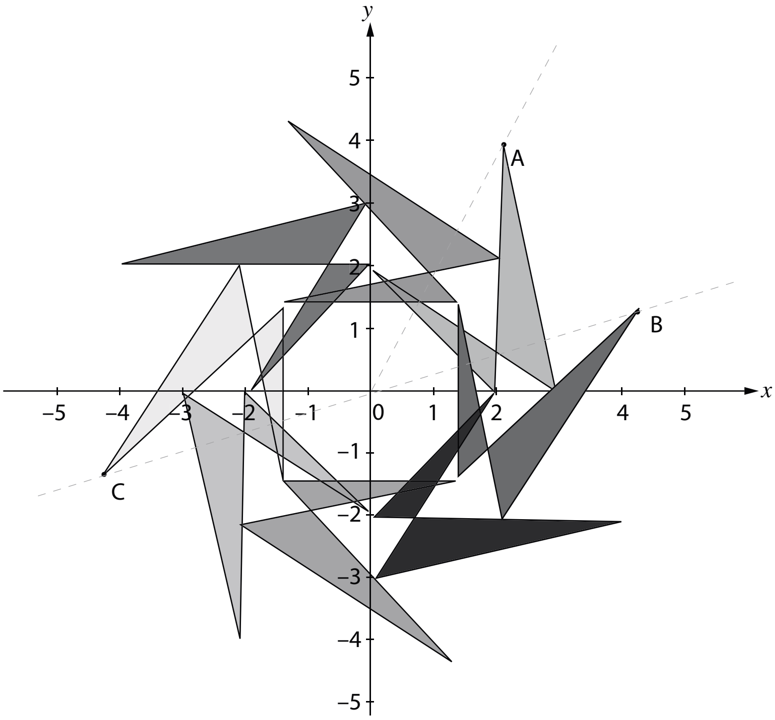
   $y' = -2{,}5.\cos 30° + (-4{,}5).\sin 30° = -4{,}4$

   $\therefore$ A″(−2,6; −4,4)

Example 5

Example

Study the logo below and answer the questions that follow:



The coordinates of A are A(2 ; 4)

1. Write down the order of rotational symmetry of the diagram (ignore the different shades of grey).
2. Write down the size of $A\hat{O}B$
3. Hence, determine the coordinates of B leaving answers in surd form
4. Determine the coordinates of C, correct to two decimal places.

Solutions

Solution

1. 8
2. $A\hat{O}B = \frac{360°}{8} \times 1 = 45°$
3. $x' = 2 \cos'(45°) - 4.\sin(-45°) = 2 \cos 45° + 4 \sin 45° = 2 \times \frac{\sqrt{2}}{2} + 4 \times \frac{\sqrt{2}}{2} = 3\sqrt{2}$

   $y' = 4 \cos(-45°) + 2 \sin'(45°) = 4 \cos 45° - 2 \sin 45° = 4 \times \frac{\sqrt{2}}{2} - 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$

   $\therefore$ B$(3\sqrt{2}; \sqrt{2})$

4. $A\hat{O}C = 135°$

$x' = 2\cos.(135°) - 4.\sin(135°) = -4{,}24 \quad (-3\sqrt{2})$

$y' = 4\cos(135°) + 2\sin(135°) = -1{,}41 \quad (-\sqrt{2})$

$\therefore$ C(−4,24; −1,41)

Preservation of shape and size of polygons after transformations

A transformation is said to be rigid if it preserves the shape and size of the original figure.

Rotations, reflections and translations are rigid transformations. These transformations produce images that are congruent to the original shape.

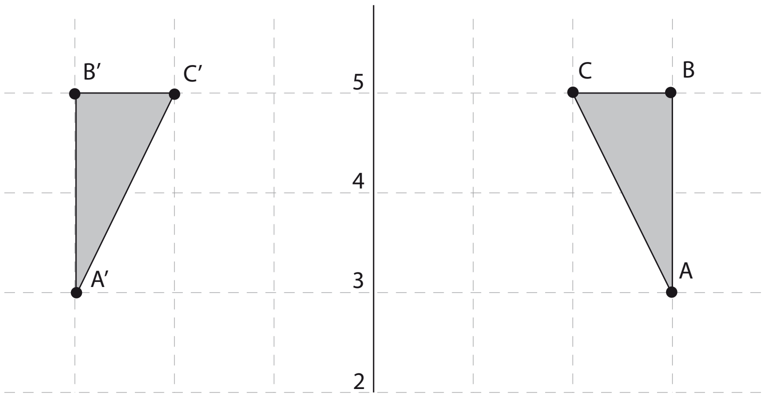
Enlargements of a polygon by a certain factor will preserve the shape but not the size. Therefore, enlargements produce images that are similar to the original shape but not congruent to the original shape.

Since they are similar, it follows that the interior angles of the polygon remain unchanged and therefore only the lengths of the sides change by the factor of the enlargement.

The examples below illustrate the above:

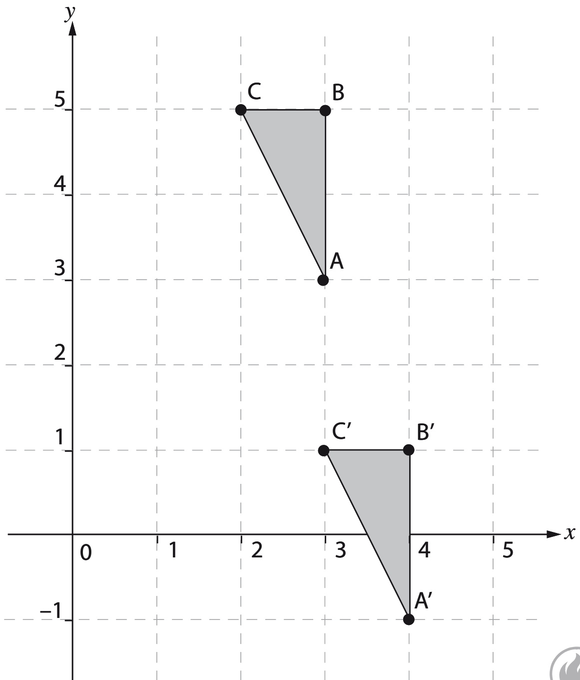
Reflections

In the example, below $\triangle$ABC is reflected about the y axis to obtain $\triangle$A′B′C′.



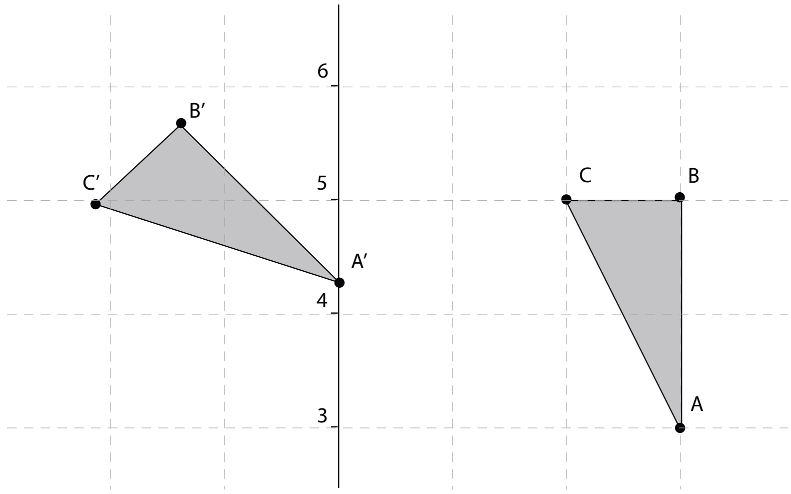
Translations

In the example, $\triangle$ABC is translated according to the rule $(x; y) \rightarrow (x + 1; y - 4)$ to obtain $\triangle$A′B′C′.



Rotations

In the example, below △ABC is rotated about the origin through an angle of 45°in an anticlockwise direction to obtain △A′B′C′.



In all the examples above, the image is congruent to the original triangle.

Notice that in all three examples, AB = A′B′; AC = A′C′; BC = B′C′.

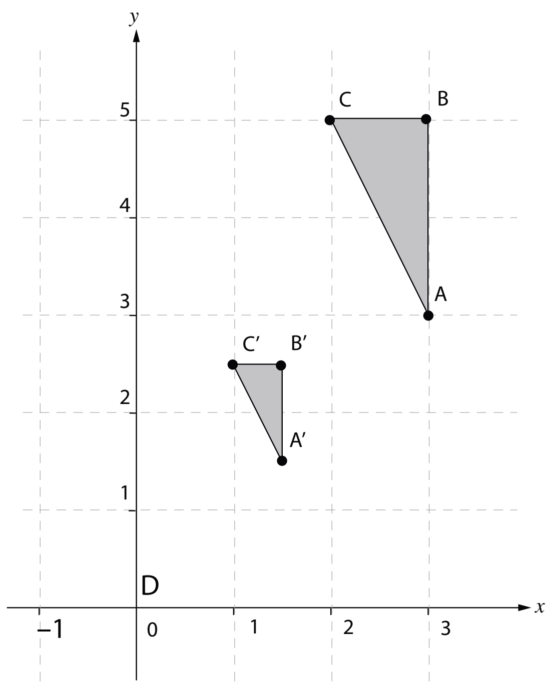
Both shape and size have remained unchanged.

Also, the area and perimeter of the two triangles are the same.

We can say, $\frac{\text{Area }\triangle ABC}{\text{Area}\triangle A'B'C'} = 1$, also $\frac{\text{Perimeter }\triangle ABC}{\text{Perimeter }\triangle A'B'C'} = 1$.

Enlargements

In the example below, △ABC is 'enlarged' by a factor of $\frac{1}{2}$ through the origin to obtain △A′B′C′.



In the example above, the image is not congruent to the original triangle. Instead △ABC is similar to △A′B′C′. That means that the lengths of AB, AC and BC have all changed by the same factor. In this case, the factor of the enlargement is $\frac{1}{2}$ and therefore we can say that each side of △A′B′C′ is half the length of the corresponding side of △ABC. For example, AB = 2 and the corresponding side A′B′ = 1.

Also, because the shapes are similar, we see that the interior angles remain unchanged. For example, $\hat{B} = 90°$ and $\hat{B}' = 90°$.

We say the shape has been preserved but not the size.

The area of the shape will change by a factor of k^2, where k is the factor of the enlargement. $(k = \frac{1}{2})$

In the above example we can say, $\frac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \frac{1}{k^2} = \frac{1}{\frac{1}{4}} = 4.$

The perimeter of the shape will change by a factor of k, where k is the factor of enlargement.

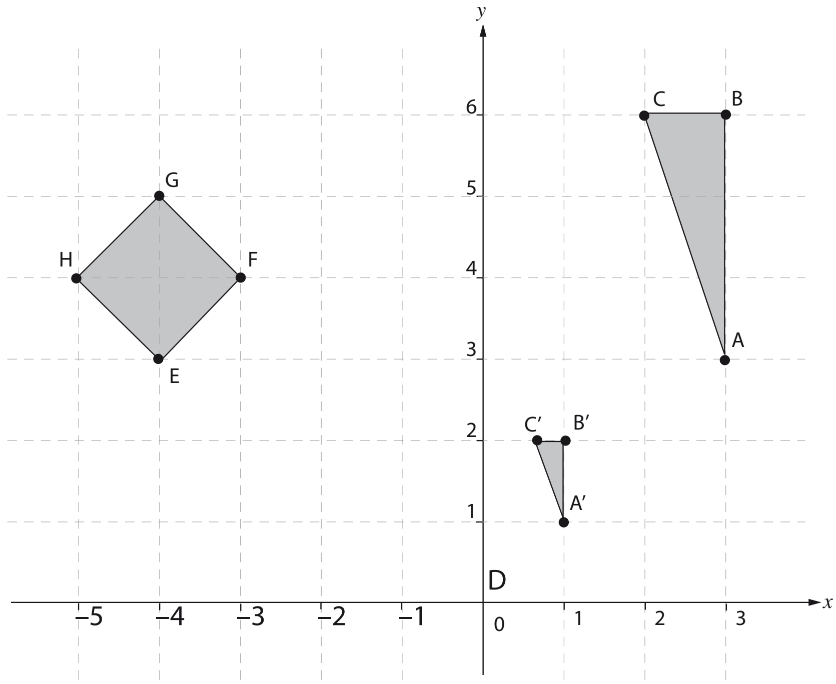
In the above example we can say, $\frac{\text{Perimeter } \triangle ABC}{\text{Perimeter } \triangle A'B'C'} = \frac{1}{k} = \frac{1}{\frac{1}{2}} = 2.$

Activity

Activity

The grade 12 examination will ask questions based on grade 10, 11 and 12 content.

1. In the diagram below, $\triangle ABC$ has been transformed to $\triangle A'B'C'$.



1.1 Describe the transformation in words

....

1.2 Give the transformation as a rule: $(x; y) \rightarrow \ldots$

....

1.3 Draw the image of EFGH if the same transformation is applied.

1.4 Complete: $\frac{\text{Area EFGH}}{\text{Area E'F'G'H'}} =$

$\frac{\text{Perimeter EFGH}}{\text{Perimeter E'F'G'H'}} =$

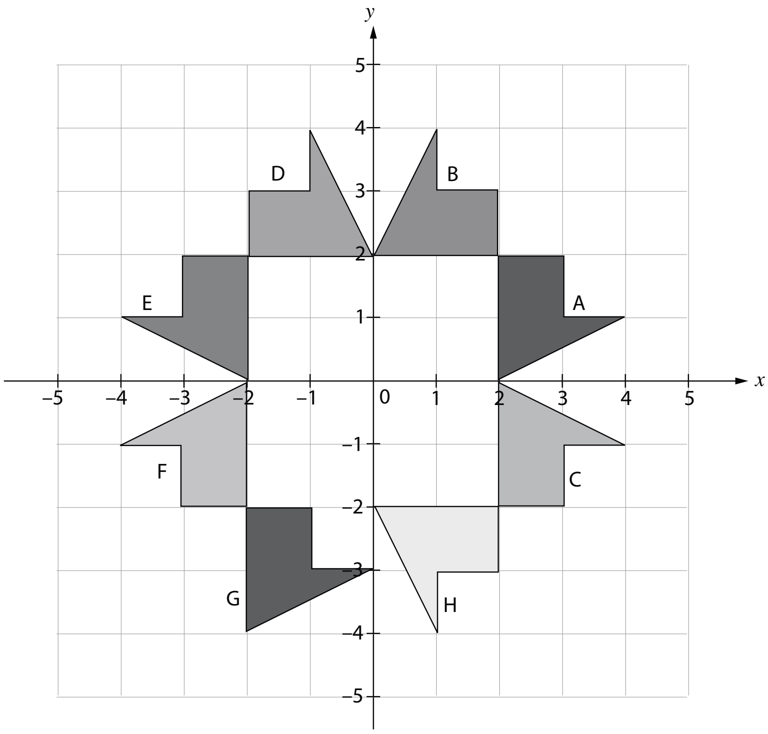
....

....

....

2. A circle with equation $x^2 + y^2 - 2x - 4y = 4$ is rotated 90° anticlockwise about the origin and then enlarged by scale factor 2. Find the new equation.

3. Study the transformations of shape A below:



In each case, for the given transformation write down the rule as $(x; y) \rightarrow \ldots$

3.1 A to B

3.2 A to C

3.3 A to D

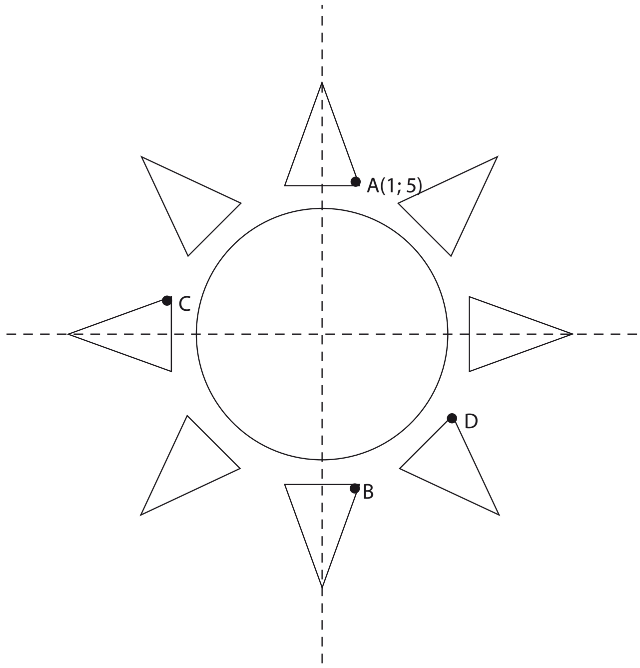
3.4 A to E

3.5 A to F

3.6 A to G

3.7 A to H

4. Study the diagram below and then answer the questions that follow:



Write down the coordinates of

4.1 B

4.2 C

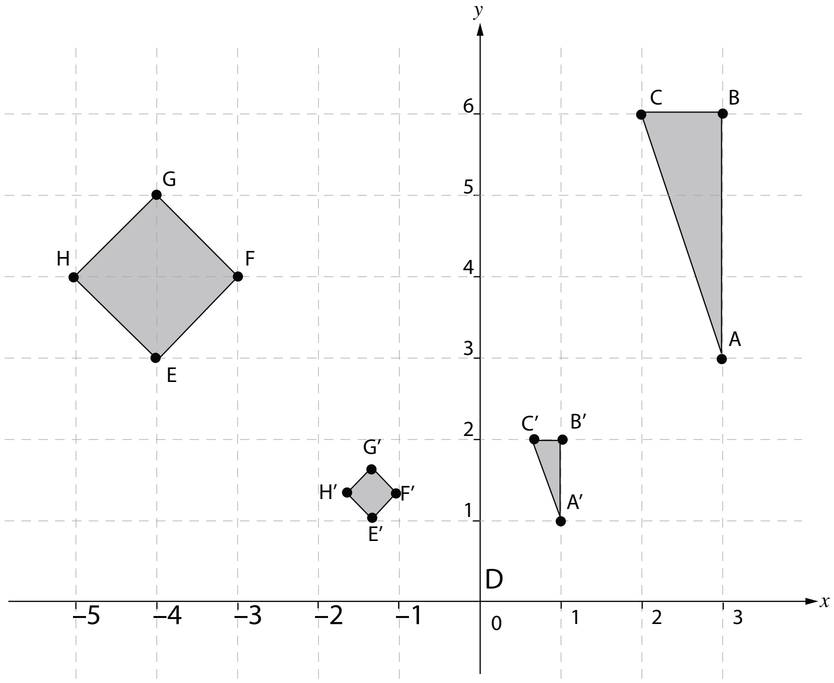
4.3 D

Solutions

1.1 Enlargement through the origin by a factor of $\frac{1}{3}$.

1.2 $(x; y) \rightarrow \left(\frac{1}{3}x; \frac{1}{3}y\right)$

1.3



1.4 $\frac{\text{Area EFGH}}{\text{Area E}'\text{F}'\text{G}'\text{H}'} = 9$ $\qquad$ $\frac{\text{Perimeter EFGH}}{\text{Perimeter E}'\text{F}'\text{G}'\text{H}'} = 3$

2. $x^2 + y^2 - 2x - 4y = 4$

$(x - 1)^2 - 1 + (y - 2)^2 - 4 = 4$

$\therefore\ (x - 1)^2 + (y - 2)^2 = 9$ $\qquad$ $\therefore$ centre (1; 2) ; $\qquad$ radius = 3

Rule for rotation through 90° anticlockwise about the origin:
$(x; y) \rightarrow (-y; x)$

Therefore, the centre of the image after a rotation of 90° is (–2; 1)

The image is then enlarged by a factor of 2.

This results in a centre of (–4; 2) and a radius of 6

$\therefore\ (x + 4)^2 + (y - 2)^2 = 36$

3.1 $(x; y) \rightarrow (y; x)$ (reflection about the line $y = x$)

3.2 $(x; y) \rightarrow (x; -y)$ (reflection about the x-axis)

3.3 $(x; y) \rightarrow (-y; x)$ (rotation 90°, anticlockwise, about the origin)

3.4 $(x; y) \rightarrow (-x; y)$ (reflection about the y-axis)

3.5 $(x; y) \rightarrow (-x; -y)$ (rotation 180°, about the origin)

3.6 $(x; y) \rightarrow (x - 4; y - 4)$ (translation, 4 units left and 4 units down)

3.7 $(x; y) \rightarrow (y; -x)$ (rotation 90°, clockwise , about the origin)

4.1 B(1; –5): reflection in the x axis

4.2 C(–5; 1): Rotation through 90° about the origin.

4.3 Rotational symmetry of order 8. Therefore $\text{A}\hat{\text{O}}\text{D} = \frac{360°}{8} \times 3 = 135°$

Rotation of 135° in a clockwise direction about the origin.

$x' = 1.\cos(-135°) - 5.\sin(-135°)$

$= \cos 135° + 5.\sin 135°$

$= -\cos 45° + 5.\sin 45°$

$= -\frac{\sqrt{2}}{2} + 5.\frac{\sqrt{2}}{2}$

$= 2\sqrt{2}$

$y' = 5.\cos(-135°) + 1.\sin(-135°)$

$= 5.\cos 135° - \sin 135°$

$= -5.\cos 45° - \sin 45°$

$= -\frac{5\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$= -3\sqrt{2}$

$\therefore D(2\sqrt{2}; -3\sqrt{2})$