LOGARITHMS

The logarithm of a number to a given base is the exponent to which that base must be raised in order to produce the number.

For example: What is the exponent that 5 must be raised to in order to produce 25?

The answer is 2 of course!

Since $5^2 = 25$.

In log form this would appear as $\log_5 25 = 2$

$N = a^k$ is called **EXPONENTIAL FORM**
$\log_a N = k$ is called **LOG FORM**
where **N > 0;** $\boldsymbol{a} > 0$ and $\boldsymbol{a} \neq 1$

Examples

Example

1. Write the exponential expression in logarithmic form

1.1 $4^3 = 64$ — $\text{base}^{\text{exponent}} = \text{number}$

$\log_4 64 = 3$

1.2 $a^b = c$

$\log_a c = b$ — ☺ Remember to put the base lower than the 'g' of log so you don't get confused

2. Write the log equation in exponential form

2.1 $\log_5 \frac{1}{125} = -3$ — $\log_{\text{base}} \text{number} = \text{exponent}$

$5^{-3} = \frac{1}{125}$

2.2 $\log_x 9 = 2$

$x^2 = 9$

3. Solve for x:

3.1 $\log_{16} 4 = x$

$\therefore 16^x = 4$ — Change to exponential form

$\therefore 4^{2x} = 4$ — Solve as a normal exponential equation

$\therefore 2x = 1$

$\therefore x = \frac{1}{2}$

3.2 $\log_x 4 = 2$

$x^2 = 4$

$x = 2$ — Normally this would be $x = \pm 2$ but because of restrictions, $x = 2$

The Log Laws

In order to make life easier when working with logs there are several laws that can be used to simplify expressions.

Law 1: $\log_k k = 1$ $\quad k > 0,\ k \neq 1$

E.g. $\log_3 3 = 1; \log_7 7 = 1$

Law 2: $\log_k 1 = 0$ $\quad k > 0,\ k \neq 1$

E.g. $\log_{\sqrt{7}} 1 = 0; \log_{\frac{1}{5}} 1 = 0$

Law 3: $\log_k x^m = m \log_k x$ $\quad k > 0, k \neq 1, x > 0$

E.g. $\log_3 3^2 = 2; \log_3 3 = 2$

Law 4: $\log_k(xy) = \log_k x + \log_k y$ $\quad k > 0, k \neq 1, x > 0, y > 0$

E.g. $\log_3(5x) = \log_3(5) + \log_3(x)$

Law 5: $\log_k \left(\frac{x}{y}\right) = \log_k x - \log_k y$ $\quad k > 0, k \neq 1, x > 0, y > 0$

$\log_3\left(\frac{5}{x}\right) = \log_3(5) - \log_3(x)$

Example

Examples

1. Use the log laws to expand as far as possible:

$\log_5 \frac{y^4}{125a}$

$= \log_5(y^4) - \log_5(125a)$ — First undo the division

$= (\log_5 y^4) - (\log_5 125 + \log_5 a)$ — Then undo the multiplication. Remember to use brackets

$= \log_5 y^4 - \log_5 5^3 - \log_5 a$ — Distribute the negative through and write numbers as exponents

$= 4 \log_5 y - 3\log_5 5 - \log_5 a$ — Now deal with the powers

$= 4 \log_5 y - 3(1) - \log_5 a$ — ☺ Remember $\log_a a = 1$

$= 4 \log_5 y - 3 - \log_5 a$

2. Express the following as the log of a single term:

$\log_2 5 - \log_2 12 + 3\log_2 2 + \log_2 3 - \log_2 5$

$= \log_2 5 - \log_2 12 + \log_2 2^3 + \log_2 3 - \log_2 5$

First use $n\log_a b = \log_a b^n$

$= \log_2 5 - \log_2 12 + \log_2 8 + \log_2 3 - \log_2 5$

$= (\log_2 5 + \log_2 8 + \log_2 3) - (\log_2 12 + \log_2 5)$

$= \log_2(5 \times 8 \times 3) - \log_2(5 \times 12)$

Now, organise all + and – signs together

$= \log_2(120) - \log_2(60)$

$= \log_2 \left(\frac{120}{60}\right)$

$= \log_2 2$

$= 1$

3. Without the use of a calculator, evaluate where possible.

$\frac{\log_3 81}{\log_3 243}$ — 81 and 243 are both powers of the base 3

$= \frac{\log_3 3^4}{\log_3 3^5}$ — Use $n\log_a b = \log_a b^n$, or $\log_a a^b = b$

$= \frac{4 \log_3 3}{5 \log_3 3}$

$= \frac{4}{5}$

4. $\frac{\log_5 16 - \log_7 4}{\log_5 4 - \log_7 2}$

$= \frac{\log_5 4^2 - \log_7 2^2}{\log_5 4 - \log_7 2}$ Simplify the logs using the log laws

$= \frac{2\log_5 4 - 2\log_7 2}{\log_5 4 - \log_7 2}$

$= \frac{2(\log_5 4 - \log_7 2)}{(\log_5 4 - \log_7 2)}$ Factorise and simplify

$= 2$

5. If $\log 2 = p$ and $\log 3 = q$, express the following in terms of p and q.

5.1 $\log(6) = \log(3 \times 2)$

$= \log(3) + \log(2)$

$= q + p$

5.2 $\log\left(\frac{2}{3}\right)$

$= \log(2) - \log(3)$

$= p - q$

5.3 $\log(0{,}9) = \log\left(\frac{9}{10}\right)$

$= \log 9 - \log 10$

$= \log 3^2 - 1$

$= 2\log 3 - 1$

$= 2q - 1$

$\log 10 = \log_{10} 10 = 1$ from $\log_k K = 1$ Note: When the base is not given like log10, the base is automatically 10. So $\log 5 = \log_{10} 5$ and $\log x = \log_{10} x\ldots$

Log equations:

In this section we are going to cover:

- Solving logarithmic and exponential equations
- Problems in every day life

Log equations can be solved through two different methods producing the same result.

Method 1: Since the log of a number to any given base is unique, it follows that if

$$\log_k A = \log_k B \Leftrightarrow A = B$$

E.g. $\log_5(x + 2) = \log_5 7$

$\therefore x + 2 = 7$

$\therefore x = 5$

Method 2: We use the relationship between logs and exponents here

$$\log_k A = B \Leftrightarrow k^B = A$$

E.g. $\log_2\left(\frac{1}{16}\right) = x$

$\therefore 2^x = \frac{1}{16}$

$\therefore 2^x = 2^{-4}$

$\therefore x = -4$

Example

Example 1

Solve for x: **Introduce log on right hand side using the rule: $\log_a a = 1$ and reduce to one log using the product law**

$\log_2(x-2) + \log_2(x-3) = 1$ **Use log laws to condense on LHS**

$\log_2(x-2)(x-3) = \log_2 2$

$(x-2)(x-3) = 2$

$x^2 - 5x + 6 = 2$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ & $x = 1$…N/A…………………………………………………

Note : $x = 1$ is not a solution since $(x-2)$ in $\log_2(x-2)$ and $(x-3)$ in $\log_2(x-3)$ are negative when $x = 1$.

Always check your solution to ensure that your answer for x does not cause the number to be a negative.

Example

Example 2

$\log_2(x-2)(x-3) = 1$ **Reduce to one log using the product law**

$2^1 = (x-2)(x-3)$ **Use method 2 and convert to the exponential form as the equation has been reduced to the desired format**

$2 = x^2 - 5x + 6$ **We now solve for x**

$0 = x^2 - 5x + 4$

$0 = (x-4)(x-1)$

$x = 4$ OR $x = 1$ N/A

Real Life applications:

There are many real life situations that involve logs and log calculations. The following examples give some real life applications of logs.

Example

Examples:

1. The energy of a radio active molecule is described by the equation below:

   $E = 6.1(2.71)^{-t}$ $t \geq 0$

   Where t is the time in seconds from the start of its existence.

1.1 What is its initial energy?

Solution

Solution

Its initial energy is when it first begins its life and therefore the time is 0. Substituting this value into the equation gives

$E = 6.1(2.71)^{-(0)}$

= 6.1(1) Anything to the power of 0 is 1

= 6.1

1.2. How long has it existed if its energy is 1.2? (rounded off to two decimal places)

Solution

Solution

Substituting this value in for the energy we can calculate the time that has passed as follows

$1.2 = 6.1(2.71)^{-t}$

$\therefore \frac{1.2}{6.1} = (2.71)^{-t}$

$\therefore (2.71)^{-t} = 0.1967213115$

$\therefore -t = \log_{2.71} 0.1967213115$

$\therefore -t = -1.6309$

$\therefore t = 1.6$ seconds

2. The measurement of noise is often described by its decibel value and is given by the equation below, where N0 represents normal noise and NI represents the noise we are trying to measure.

Noise (in decibels) = $20\log\left(\frac{NI}{N0}\right)$

2.1 How much noise is there in decibels if we measure 4 times more noise than usual? (Rounded off to two decimal places)

Solution

Solution

This means that NO = 4NI and therefore $\frac{NO}{NI}$

$= \frac{4NI}{NI}$

= 4

Thus substituting the values into the equation gives

Noise (in decibels) = 20log(4)

= 12.04decibels Using a calculator, rounding to 2d.p.

2.2 If the noise measured gives 100-decibels how much larger is the measured noise than normal noise?

Solution

Solution

As the question asks how much larger the measured noise is to regular noise we can say that for the time being that it is 'x' times larger where x is the value we are looking for. Therefore the noise itself is x times N0.

The expression of $\frac{NI}{NO}$ is therefore = $\frac{xN0}{N0}$

Which actually gives: x

We can then use the equation given, substituting the value of 100 in on the right for the value of decibels and the value of x for the argument of the log

$100 = 20\log(x)$ Dividing both sides by 20

$5 = \log(x)$ since: $\log(x) = \log10(x)$ [base of 10]

$x = 10^5$ Taking the log back to exponential form

$x = 100000$

Example

Example

3. A population of rabbits increases according to the following equation, where n represents the number of rabbits and t represents the time in years.

$n = 24(4.5^t)$

3.1 How many rabbits were there to start with?

3.2 How long will it take for the population to reach 4640?

Solution

Solution:

3.1 At the start t = 0 (time equals 0)

$\therefore n = 24(4{,}5^0)$

$\therefore n = 24$

3.2 This means that n=4640. So substitute into the equation and solve for t

$\therefore 4640 = 24(4.5^t)$

$\therefore \frac{4640}{24} = 4{,}5^t$

$\therefore 4.5^t = 193{,}\dot{3}$

$\therefore t = \log_{4.5} 193{,}\dot{3}$

$\therefore t = 3{,}5$ years

Activity

Activity

1. Write in log form:

1.1 $3^4 = 81$

....

1.2 $25^6 = 244{,}14$

....

2. Write in exponential form:

2.1 $\log_5 125 = 3$

....

2.2 $\log_{\frac{3}{2}} \frac{27}{8} = 3$

....

3. Solve for x:

3.1 $\log_x 256 = 4$

....

....

....

....

3.2 $\log_8 2 = x$

4. Use the log laws to simplify:

4.1 $\log_a 6a - \log_a 3 - \log_a 2$

4.2 $\log_{10}\frac{100}{4} + 2\log_{10} 2$

5. If $\log_3 4 = k$ write in terms of k: $\log_3 \frac{16}{3}$

6. Solve for x:

6.1 $\log_2(x - 8) + \log_2(x + 2) = \log_2 24$

6.2 $\log_3(x - 2) = 1$

7. The intensity measure of a television is given by:

$I = 36\log\left(\frac{A}{Ao}\right)$

Where A represents the measured amplitude of the television's intensity and Ao represents the normal amplitude of light intensity.

7.1 What is the light intensity if the television emits light at an intensity of 6.5 times that of normal light?

7.2 How much larger is the intensity of the television if it gives an intensity reading of 13,2?

Solutions

1.1 $\log_3 81 = 4$

1.2 $\log_{25} 244.14 = 6$

2.1 $5^3 = 125$

2.2 $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$

3.1 $x^4 = 256$

$x = \sqrt[4]{256}$

$= 4$

3.2 $8^x = 2$

$(2^3)^x = 2^1$

$3x = 1$

$x = \frac{1}{3}$

4.1 $\log_a 6a - \log_a 3 - \log_a 2$

$= \text{Log}_a \frac{6a}{(3)(2)}$

$= \log_a a$

$= 1$

4.2 $\log_{10} 100 - \log_{10} 4$

$= \log_{10} 10^2 - \log_{10} 2^2$

$= 2 - 2\log_{10} 2 = 2 - 2\log 2$

5. $\log_3 16 - \log_3 3$

$= \log_3 4^2 - 1$

$= 2\log_3 4 - 1$

$= 2k - 1$

6.1 $\log_2 (x-8)(x+2) = \log_2 24$

$(x-8)(x+2) = 24$

$x^2 - 6x - 16 = 24$

$x^2 - 6x - 40 = 0$

$(x-10)(x+4) = 0$

\` $x = 10$ AND $x = -4$ N/A since $x = -4$ causes the numbers of the logs to be negative

6.2 $\log^3(x - 2) = 1$

$\therefore 3^1 = x - 2$

$\therefore 5 = x$

7.1 This means that the measured light intensity A= 6,5Ao

$\therefore$ Substitute into the equation and solve for I

$\therefore \ I = 36\log\left(\frac{6{,}5Ao}{Ao}\right)$

$\therefore \ I = 36\log(6.5)$

$\therefore \ I = 29{,}3$ (1d.p.)

7.2 Substituting the value of 13.2 for I and working out the ratio of $\frac{A}{Ao}$

$\therefore \ 13.2 = 36\log\left(\frac{A}{Ao}\right)$

$\therefore \ \frac{13.2}{36} = \log\left(\frac{A}{Ao}\right)$

$\therefore \ \log\left(\frac{A}{Ao}\right) = 0{,}36\dot{6}$

$\therefore \ \frac{A}{Ao} = 10^{0.36\dot{6}}$

$\therefore \ \frac{A}{Ao} = 2{,}3$ (1d.p.)

This means that the measured intensity is 2,3 times bigger than the normal amplitude.